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The Mathematics of Sound: Fourier Analysis and Chord Detection in Music

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Chronicle

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Abstract

This paper develops a representation of a sequence in terms of sample of its short-time Fourier transform and synthesis of the original sequence. Here, the problem of chord detection where one wishes to identify play chords in a music life is discussed. By investigation, the DFT alongside an application in music processing. Finally, the method of Discrete Fourier transform (DFT) is considered as special cases of this representation.

Keywords: chord detection, Fourier series, pitches, fast Fourier transform, discrete Fourier transform, continuous Fourier transform, dirac delta function.

INTRODUCTION

The Fourier transform plays a fundamental role in the analysis of signals and linear-time invariant systems. In modern world, problems which are traditionally analog signals are discretized to enable computer analysis. A basic tool used by mathematicians, engineers, and scientists in this context is discrete Fourier trans-form which allows us to analyze individual's frequency components of digital signals. (Nathan & Deanna, 2014) used Fourier transform methods in analyzing mu-sic processing. Music comes from Greek word" mausike" which means" art of muses". It is a form of art, an expression of emotions through harmonic frequencies. The vast majority of music is governing by special rules for beat, time, pitch, rhythm, and harmony. These rules, and sample they attract mathematicians, engineers, and statisticians (Alexander & Daniel,2003) to develop algorithms elements of songs. The problem discussed in paper is" chord detection", where one wants to identify play chords within a music file. With the capability to determine the harmonic structure of songs, massive data bases which would be bloom choice for statistical analysis can be built. Anyone performing a task must be highly educated in music theory and would likely take hours to complete the annotation of one song, but an average computer can do a task with reasonable accuracy in just a second. This paper explores the mathematics behind such a program and explain how one can use such tools to directly analyze audio files.

Music Theory

Music Application Sound waves are one type of waves that can be analyzed using Fourier series, allowing for different aspects of music to be analyzed using this method. Musical instruments produce sound as a result of the vibration of a physical object such as a string on a violin, guitar, or piano, or a column of air in a brass or woodwind instrument (Christopher, 2002). This vibration causes a periodic variation in air pressure that is heard as sound. The period T is the length of time before the signal repeats, and the frequency f1 equal to 1 is the fundamental frequency or pitch of the note produced. When this vibration occurs in a musical instrument, not only is the frequency f_1 produced, but other frequencies are produced as well. These frequencies are integer multiples of the fundamental frequency f_1. Thus, in addition to f_1, there are frequencies f_2, which equals 2f_1,f_3, which equals 3f_1,f_4, and so on. The frequency f_1 is called the fundamental, and the frequencies 〖f〗_2, f_3, f_4, etc. are called the harmonics.

Notations and Definitions

Definition 3.1. Pitches and Scales Pitch is an indication of a sound perceived frequency, from low to high. For instance, the tuning note for a symphony orchestra is A5 which is a standardized frequency of 880Hz. In this notation A5, A indicates the chorma or equality of a note while 5 describes octave or height. A" scale" is any set of musical notes ordered by basis frequency or pitch. As we follow, the pitches of scale from bottom to top, we start an end on same note one octave apart. Doubling in frequency of sound wave results in an increase in one-octave. In chromatic scale, 12 chroma are ordered over an octave. These twelve notes are spaced logarithmically over the octave. We use recursive sequence as Lee (2006) proposed an automatic chord recognition method using an enhanced pitch class profile to describe

$$\boldsymbol{\pi} = \frac{\mathbf{2^{1/2}}}{\boldsymbol{\Pi_{\iota-1}}}$$

where π: frequency of one pitch and Π_(ı-1): frequency of previous pitch.We can hear the chromatic scale by striking every white and black key of a piano in order up an active or imagine it by a scale. We can hear the chromatic scale by striking even white and black key of a piano in order up an active or imagine it by a scale.

Definition 3.2. Interval In music theory, an interval is difference in pitch between two sands. An interval may be described as horizontal, linear, or melodic if it refers to simultaneously sounding tones such as in a chord. Scales are defined by a sequence of their intervals with the condition that the total sums of steps must equal to 12. This guaranties that we start and end on the same chroma known as the root R. Major, minor, argumented, and diminished are four prevalent scales in music theory. The intervals that describe these scales can be found in table 1.Table 1: Internal construction for four core scales

Scale"color"	Defining Steps
Chromatic	RHHHHHHHHHHHHR
Major	RWWHWWWHR
Minor	RWHWWHWWR
Diminished	RHWHWHWHWR
Augmented	RAHAHAHR

The table is used for selecting any starting any note for the root and then using the intervals to construct the remaining notes. For instance, a C minor (C_m) scale is C, D, E, F, G, A, B, C.

Definition 3.3. Triads and Chords A chord is any combination of three or more pitch classes that sound simultaneously. A three-note chord whose pitch classes can be arranged as third is called a triad. The pitch classes of a triad will always sit next to each other. These triads describe the major, minor, diminished, augmented chords in our model. While triads are useful way to understand the tonal structure of music.

Chord is created by taking a major triad and adding a note a minor seventh above the root. For example, a dominant seventh chord on C contains the notes C-E-G-Bb. (Bb is a minor 7/10 semitones above C). The dominant is used a lot in blue music. In music, the triad's members, from lowest pitched tone to highest, are called: the root. The third its interval above the root being a minor third and a major third. The theory behind the dominant seventh is the consequence of the theory of musical modes, we refer the interested reader to (Laitz, 2011) for more information.

Definition 3.4. Chord Inversion An inverted chord means that you have move the root of the chord to some upper position, leaving the note other than the root as the lowest sounding note. It is a great device that will add colors to your musical palette. With the root on the bottom you get this triad in it's most stable position. The concept of root has been extended for the description of intervals of two notes. The interval can either be analyzed as formed from stacked third, fifth, seventh, etc (odd). And is how note considered as root, or as an inversion of the same: second (inversion of seventh), fourth (inversion of fifth), sixth (inversion of third), etc (even) in which cases upper note is the root. Here we described seven chord families and twenty-one roots, giving the total of 147 different possible chords, when chords involve a seventh, one can perform three inversions. Including inversions, the total number of chords is C = (21 roots) [(4 triads) (2 inversions) +(3 chords) (3 inversions)] = 357.

Definition 3.5. Chord Detection It is a special form of music transcriptions that captures only harmonic properties of the audio signals. It is particularly interesting as chords are comparatively simple and stable structures, and as the same time completely describes a piece of music in terms of occurring harmonics. In this article we will focus on a chord recognition model (Sheh & Ellis, 2003).

Definition 3.6. The Chromagram In music, the term chroma feature or Chromagram closely relates to the twelve different pitch classes. Chroma-based features, which are also referred to as" pitch class profiles", are a powerful tool for analyzing music whose pitches can be meaningfully categorized (often into twelve categories) and whose tuning approximates to the equal-tempered scale. One main property of chroma features is that they capture harmonic and melodic characteristics of music, while being robust to changes in timbre and instrumentation. The underlying observation is that humans perceive two musical pitches as similar in color if they differ by an octave. Based on this observation, a pitch can be separated into two components, which are referred to as tone height and chroma (Shepard, 1964). Assuming the equal-tempered scale, one considers twelve chroma values represented by the set C,C(#),D,D(#),E,F,f(#),G,G(#),A,A(#),B that consists of the twelve pitch spelling attributes as used in Western music notation. Note that in the equal-tempered scale different pitch spellings such C(#) and D(ь) refer to the same chroma. Enumerating the chroma values, one can identify the set of chroma values with the set of integers 1,2,...,12, where 1 refers to chroma C, 2 to C(#), and so on. A pitch class is defined as the set of all pitches that share the same chroma. Since the DFT is a sum of indexed values we can express equation in matrix form as the linear equation:

$$\boldsymbol{F} = \boldsymbol{Wf}$$

where f is the vector in the time domain of length N, F is the output in the frequency domain, and W is the non singular matrix. We know that W must be non singular as one can readily verify that the columns are orthogonal. Since W is non singular, we may express the inverse DFT by: f = W^(-1)F.

Using this method of matrix multiplication, we can calculate the DFT in O(n^2) time where n is the length of the input vector. However, with sampled music, the inputs are extremely high dimensional and we would like to find a method that computes the DFT much faster.

Definition 3.7. Spectrogram A spectrogram is a visual representation of the spectrum of frequencies of a signal as it varies with time. When applied to an audio signal, spectrograms are sometimes called sonographs, voice prints, or voice grams. When the data is represented in a 3D plot they may be called waterfalls.

They are used extensively in the fields of music, sonar, radar, and speech processing, (Flanagan, 1972) seismology, and others. Spectrograms of audio can be used to identify spoken words phonetically, and to analyse the various calls of animals.

A spectrogram can be generated by an optical spectrometer, a bank of band-pass filters, by Fourier transform or by a wavelet transform (in which case it is also known as a scaleogram) (Sejdic et al.,2008).

Definition 3.8. Sinusoids Electronic music is usually made using a computer, by synthesizing or processing digital audio signals. These are sequences of numbers,

$$\ldots, x[n-1], x[n], x[n+1], \ldots$$

where the index n, called the sample number, may range over some or all the integers. A single number in the sequence is called a sample. An example of a digital audio signal is the Sinusoid:

$$x[n] = a\cos(\omega n + \varphi)$$

where a is the amplitude, ω is the angular frequency, and φ is the initial phase. The phase is a function of the sample number n, equal to ωn + φ. The initial phase is the phase at the zeroth sample (n = 0).

A digital audio signal, showing its discrete-time nature (part a), and idealized as a continuous function (part b). This signal is a (real-valued) sinusoid, fifty points long, with amplitude 1, angular frequency 0.24, and initial phase zero. fig Sinusoids play a key role in audio processing because, if you shift one of them left or right by any number of samples, you get another one. This makes it easy to calculate the effect of all sorts of operations on sinusoids. Our ears use this same special property to help us parse incoming sounds, which is why sinusoids, and combinations of sinusoids, can be used to achieve many musical effects.

Digital audio signals do not have any intrinsic relationship with time, but to listen to them we must choose a sample rate, usually given the variable name R, which is the number of samples that fit into a second. The time t is related to the sample number n by Rt = n, or t = n/R. A sinusoidal signal with angular frequency ω has a real-time frequency equal to

$$\frac{\omega R}{f\,2\pi}$$

in Hertz (i.e., cycles per second), because a cycle is 2π radians and a second is R

samples. We define a sinusoid as a function of the form:

$$x(t) = A\sin(2\pi vt + \varphi).$$

$$A = \text{amplitude};$$

ν = radian frequency(rad/sec);

2πv = frequency(Hz);

t = time(s);

ϕ = initial phase(radians);

2πvt + φ = instantaneous quadphase(radians).

Fourier transforms are built on the complex properties of sinusoids which follow from Euler's identities

$$〖\exp〗^{\imath\theta} = \cos(\theta) + \imath\sin(\theta), \quad (3.1)$$

$$\exp \pm \imath 2\pi\nu x = \cos(2\pi\nu x) \pm \imath \sin(2\pi\nu x), \quad (3.2)$$

the latter being the form most relevant to audio signal processing.

INTRODUCTION TO FOURIER TRANSFORM

Notations, Definitions, Examples, Proposition, and Theorems

Definition 5.1. Frequency is the number of occurrences of a repeating event per unit of time.ref 1 It is also referred to as temporal frequency, which emphasizes the contrast to spatial frequency and angular frequency. Frequency is measured in units of hertz (Hz) which is equal to one occurrence of a repeating event per second. The period is the duration of time of one cycle in a repeating event, so the period is the reciprocal of the frequency.ref 2.

For cyclical processes, such as rotation, oscillations, or waves, frequency is defined as a number of cycles per unit time. In physics and engineering disciplines, such as optics, acoustics, and radio, frequency is usually denoted by a Latin letter f or by the Greek letter v. The relation between the frequency and the period, T, of a repeating event or oscillation is given by f =1/T.

Definition 5.2. Time Domain and Frequency Domain Time domain refers to the analysis of mathematical functions, physical signals or time series of economic or environmental data, with respect to time. In the time domain, the signal or function's value is known for all real numbers, for the case of continuous time, or at various separate instants in the case of discrete time. An oscilloscope is a tool commonly used to visualize real-world signals in the time domain. A time-domain graph shows how a signal changes with time, whereas a frequency-domain graph shows how much of the signal lies within each given frequency band over a range of frequencies.

Frequency domain refers to the analysis of mathematical functions or signals with respect to frequency, rather than time. ref1 Put simply, a time-domain graph shows how a signal changes over time, whereas a frequency-domain graph shows how much of the signal lies within each given frequency band over a range of frequencies. A frequency-domain representation can also include information on the phase shift that must be applied to each sinusoid in order to be able to recombine the frequency components to recover the original time signal.

Definition 5.3. Fourier Series In mathematics, a Fourier series is a periodic function composed of harmonically related sinusoids, combined by a weighted sum-motion. As such, the summation is a synthesis of another function. As such, the summation is a synthesis of another function. The discrete-time Fourier transform is an example of

Fourier series. The process of deriving the weights that describe a given function is a form of Fourier analysis. For functions on unbounded intervals, the analysis and synthesis analogies are Fourier transform and inverse transform. Consider a real-valued function, s(x), that is integral on an interval of length P, which will be the period of the Fourier series. Common examples of analysis intervals are:

$x \in [0,1]$, and $P = 1$. $x \in [-\pi, \pi]$, and $P = 2\pi$

The analysis process determines the weights, indexed by integer n, which is also the number of cycles of the nth harmonic in the analysis interval. Therefore, the length of a cycle, in the units of x, is P/n . And the corresponding harmonic frequency is n/P.

The n^th harmonics are:

$$Sin\left(\frac{2\pi xn}{P}\right) \text{ and } Cos\left(\frac{2\pi xn}{P}\right)$$

and their amplitudes (weights) are found by integration over the interval of length P . The actual Fourier Series, sine-cosine form is:

$$S_n(x) = \frac{a_0}{2} + \sum_{n=1}^{N}\left(a_n Cos\left(\frac{2\pi xn}{P}\right) + b_n Sin\left(\frac{2\pi xn}{P}\right)\right)$$

Definition 5.4. Fourier Transform The Fourier transform (FT) decomposes a function (often a function of the time, or a signal) into its constituent frequencies. A special case is the expression of a musical chord in terms of the volumes and frequencies of its constituent notes. The term Fourier transform refers to both the frequency domain representation and the mathematical operation that associates the frequency domain representation to a function of time. The Fourier transform of a function of time is itself a complex-valued function of frequency, whose magnitude (modulus) represents the amount of that frequency present in the original function, and whose argument is the phase offset of the basic sinusoid in that frequency. The Fourier transform is not limited to functions of time, but the domain of the original function is commonly referred to as the time domain.

There is also an inverse Fourier transform that mathematically synthesizes the original function from its frequency domain representation. A special case is the expression of a musical chord in terms of the volumes and frequencies of its constituent notes. The term Fourier transform refers to both the frequency domain representation and the mathematical operation that associates the frequency domain representation to a function of time.

The Fourier transform f is traditionally denoted (f)̂ by adding a circumflex to the symbol of the function. There are several common conventions for defining the Fourier transform of an integral function

$f: \mathbb{R} \to \mathbb{C}$

(Bailey& Swarztrauber,1994)., (Boashash, 2003). One of them is

$$f(\xi) = \int_{-\infty}^{\infty} f(x)\exp^{-2\pi i x\xi} dx, \text{ for any real number } \xi$$

Definition 5.5. Discrete Fourier Transform (DFT) In mathematics, the dis-crete Fourier transform (DFT) converts a finite sequence of equally-spaced samples of a function into a same-length sequence of equally-spaced samples of the discrete-time Fourier transform (DTFT), which is a complex-valued function of frequency.

An inverse DFT is a Fourier series, using the DTFT samples as coefficients of com-plex sinusoids at the corresponding DTFT frequencies. In image processing, the samples can be the values of pixels along a row or column of a raster image. The DFT is also used to efficiently solve partial differential equations, and to perform other operations such as convolutions or multiplying large integers. Since it deals with a finite amount of data, it can be implemented in computers by numerical algorithms or even dedicated hardware. Prior to its current usage, the "FFT" ini-tialism may have also been used for the ambiguous term "finite Fourier transform". In the discrete time case, the data to be transformed could be broken up into chunks or frames (which usually overlap each other, to reduce artifacts at the boundary). Each chunk is Fourier transform, and the complex result is added to a matrix, which records magnitude and phase for each point in time and frequency. This can be expressed as:

$$\mathbf{X[n](m,\omega)} = \sum_{\mathbf{n=-\infty}}^{\infty} \mathbf{[n]\omega[n - \omega]\, exp^{-j\omega n}}$$

likewise, with signal x[n] and window ω[n]. In this case, m is discrete and ω is continuous, but in most typical applications the STFT is performed on a computer using the Fast Fourier transform, so both variables are discrete and quantized. The magnitude squared of the STFT yields the spectrogram representation of the Power Spectral Density of the function:

$$\boldsymbol{X[n](m,\omega)} \equiv |\boldsymbol{X(m,\omega)}|^2$$

Example 5.6. Using the DFT to adjust frequencies in sound Since the DFT coefficients represent the contribution in a sound at given frequencies, we can listen to the different frequencies of a sound by adjusting the DFT coefficients. Let us first see how we can listen to the lower frequencies only. As explained, these correspond to DFT-indices n in [0, L]∪[N −L, N −1]. In MATLAB indices between L + 2 and N − L. We can now perform inverse DFT to recover the sound signal with these frequencies eliminated. With the help of the DFT implementation all this can be achieved with the following code:

```
y=DFTImpl(x);
y((L + 2): (N - L)) = zeros(N - (2 * L + 1),1);
newx = IDFTImpl(y)
```

Definition 5.7. Fast Fourier Transform FFT The Fourier transform (FT) de-composes a function (often a function of the time, or a signal) into its constituent frequencies. A special case is the expression of a musical chord in terms of the volumes and frequencies of its constituent notes. The term Fourier transform refers to both the frequency domain representation and the mathematical operation that associates the frequency domain representation to a function of time. The Fourier transform of a function of time is itself a complex-valued function of frequency, whose magnitude (modulus) represents the amount of that frequency present in the original function, and whose argument is the phase offset of the basic sinusoid in that frequency. The Fourier transform is not limited to functions of time, but the domain of the original function is commonly referred to as the time domain. There is also an inverse Fourier transform that mathematically synthesizes the original function from its frequency domain representation.

The Fourier transform of a function f is traditionally denoted f̂, by adding a circumflex to the symbol of the function. There are several common conventions for defining the Fourier transform of an integral function f:R⟶C . One of them is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{f(x)exp^{-2\pi ix\xi}\,dx},$$

For any real number ξ

Theorem 5.8. Fast Fourier Transform FFT algorithm Let y = FN x be the N-point DFT of x with N an even number. From any integer n in the interval [0, N/2 − 1] the DFT y of x is then given by

$$_n = \frac{1}{\sqrt{2}}(F_{N/2}x^e)_n + \exp^{-2\pi \iota n/N}(F_{N/2}x^0)_n),$$

$$y_{N/2+n} = \frac{1}{\sqrt{2}}((F_{N/2}x^e)_n - \exp^{-2\pi \iota n/N}(F_{N/2}x^0)_n)$$

where x^e, x^0 are the sequences of length N/2 consisting of the even and odd samples of x respectively. In other words,

$$(x^e)k = x_{2k} for\ 0 \le k \le N/2-1,$$

$$(x^0)k = x_{2k+1} for\ 0 \le k \le N/2 - 1$$

Proof. Suppose first that 0 ≤ n ≤ N/2 − 1. We start by slitting the sum in the expression for the DFT into even and odd integers,

$$y_n = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} x_k\, exp^{-2\pi ink/N}$$

$$y_n = \frac{1}{\sqrt{N}}\sum_{k=0}^{N/2-1} x_{2k}\, exp^{-2\pi in2k/N} + \frac{1}{\sqrt{N}}\sum_{k=0}^{N/2-1} x_{2k+1}\, exp^{-2\pi in(2k+1)/N}$$

$$y_n = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{N/2}}\sum_{k=0}^{N/2-1} x_{2k}\, exp^{-2\pi ink/(N/2)} + \frac{1}{\sqrt{N}}\frac{1}{\sqrt{N/2}}\sum_{k=0}^{N/2-1} x_{2k+1}\, exp^{-2\pi in/(N/2)}$$

$$y_n = \frac{1}{\sqrt{2}}(F_{N/2}x^e)_n - \frac{1}{\sqrt{2}}exp^{-2\pi in/N}(F_{N/2}x^0)_n$$

this concludes the proof.

Definition 5.9. Continuous Fourier Transform CFT The Fourier transform pair in the most general form for a continuous and a periodic time signal x(t) is

The spectrum X(jω) is expressed as a function of (jω) because the spectrum can be treated as the Laplace Transform of the signal X(s) evaluated along the imaginary axis (σ = 0):

$$X(j\omega) = X(s)|_{\delta=j\omega}\ (\textbf{in general}, s = \sigma + j\omega)$$

As this notation is closely related to the system analysis concepts such as Laplace transform and transfer function H(s), it is preferred in the field of system design nd control. However, in practice, it is more convenient to represent the frequency of a signal by f = ω/2π in cycles/ second or Hertz(Hz, KHz, MHz, GHz,etc), instead of ω in radian/second. Replacing ω by 2πf, we can also express the spectrum as X(j2πf) or simply X(f) in this alternative representation:

$$X(\boldsymbol{f}) = \int_{-\infty}^{\infty} \boldsymbol{x}(\boldsymbol{t})\boldsymbol{exp}^{-\mathrm{j}2\pi\mathrm{ft}}\boldsymbol{dt}$$

$$\boldsymbol{x}(\boldsymbol{t}) = \int_{-\infty}^{\infty} \boldsymbol{X}(\boldsymbol{f})\boldsymbol{exp}^{\mathrm{j}2\pi\mathrm{ft}}_{\boldsymbol{\xi}}\boldsymbol{dt}$$

Definition:2 Provided that the integrals exist, the following holds:

$$\hat{\boldsymbol{f}}(\boldsymbol{\xi}\ ) = \int_{-\infty}^{\infty} \boldsymbol{f}(\boldsymbol{x})\boldsymbol{exp}^{-\pi\mathrm{x}\xi}\ \boldsymbol{dx}$$

$$\boldsymbol{f}(\boldsymbol{x}) = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\pi}}\int_{-\infty}^{\infty} \hat{\boldsymbol{f}}(\boldsymbol{\xi}\ )\boldsymbol{exp}^{-\pi\mathrm{x}\xi}\ \boldsymbol{d}$$

The top equation defines the Fourier transform (FT) of the function f, the bottom equation defines the inverse Fourier transform of f ˆ. f and f ˆ are in general complex functions.

Example 5.10. Another common variant is

$$\boldsymbol{H}(\boldsymbol{f}\ \ = \int_{-\infty}^{\infty} \boldsymbol{h}(\boldsymbol{t})\boldsymbol{exp}^{-\mathrm{i}2\pi\mathrm{ft}}\boldsymbol{dx}$$

$$\boldsymbol{h}(\boldsymbol{t}) = \int_{-\infty}^{\infty} \boldsymbol{H}(\boldsymbol{f})\boldsymbol{exp}^{-\mathrm{i}2\pi\mathrm{ft}}\ \boldsymbol{d}\mathbf{f}$$

where the need for a normalization factor is eliminated by the substitution ω = 2πf . The different definitions above are related by trivial substitution or scaling of the variables. It is however important to know which definition is

used in a particular application. The special case of x=0 (or t=0) and ξ = 0 (or ω = 0 are often useful "test cases", then it gives:

$$\hat{\mathbf{f}}(\mathbf{0}\ ) = \int_{-\infty}^{\infty} \mathbf{f}(\mathbf{x})\mathbf{dx}$$

$$\boldsymbol{f}(\mathbf{0}) = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\pi}}\int_{-\infty}^{\infty} \hat{\boldsymbol{f}}(\boldsymbol{\xi}\ )\boldsymbol{d}\ \boldsymbol{\xi}$$

Definition 5.11. Dirac Delta Function The Dirac delta function δ is a generalized function or distribution introduced by the physicist Paul Dirac. It is used to model the density of an idealized point mass or point charge as a function equal to zero everywhere except for zero and whose integral over the entire real line is equal to one (Aratyn& Rasinariu, 2006), (Arfken& Weber 2000). As there is no function that has these properties, the computations made by the theoretical physicists appeared to mathematicians as nonsense until the introduction of distributions by Laurent Schwartz to formalize and validate the computations. As a distribution, the Dirac delta function is a linear functional that maps every function to its value at zero. The Kronecker delta function, which is usually defined on a discrete domain and takes values 0 and 1, is a discrete analog of the Dirac delta function. The Dirac delta can be loosely thought of as a function on the real line which is zero everywhere except at the origin, where it is infinite,

$$\boldsymbol{\delta}(\mathbf{t}) = \begin{cases} \infty\ \mathbf{if}\ \mathbf{t} = \mathbf{0} \\ \mathbf{0}\ \mathbf{if}\ \mathbf{t} \neq \mathbf{0} \end{cases}$$

Where $\int_{-\infty}^{\infty} \boldsymbol{\delta}(\mathbf{t})\boldsymbol{dx} = \mathbf{1}$ **(5.1)**

This is merely a heuristic characterization. The Dirac delta is not a function in the traditional sense as no function defined on the real numbers has these properties. The Dirac delta function can be rigorously defined either as a distribution or as a measure.

Relation Of the DFT Fourier Series

We now show that the DFT of a sampled signal x(n) (of length N), is proportional to the Fourier series coefficients of the continuous periodic signal obtained by repeating and interpolating x. More precisely, the DFT of the N samples comprising one period equals N times the Fourier series coefficients. To avoid aliasing upon sampling, the continuous-time signal must be band limited to less than half the sampling rate this implies that at most N complex harmonic components can be nonzero in the original continuous-time signal.

If x(t) is band limited to ωT ∈(−π, π), it can be sampled at intervals of T seconds without aliasing the way to sample a signal inside an integral expression such as to multiply it by a continuous-time impulse train. Fourier series with coefficients given by

$$\boldsymbol{X(\omega_k) = \frac{1}{P}\int_0^P x(t)\exp^{-jw^k t}\, dt, k = 0, \pm 1, \pm 2, \ldots}$$

$$\boldsymbol{\Psi_T(t) = T\sum_{-}^{\infty} \delta(t - nT)}$$

where δ(t) is the continuous-time impulse signal. The continuous-time Fourier series of the sampled periodic signal x(nT)is required. Thus, replacing x(t) in Eq.(6.1) by

$$\boldsymbol{x_s(\omega_k) = x(t).\Psi T\,(t)}$$

By the sifting property of delta functions the Fourier series of 〖 x〗 _s is

$$\boldsymbol{X_s(\omega_k) = \frac{1}{P}\int_0^P x_s(t)\exp^{-j\omega_k t}\, dt,} \tag{6.4}$$

$$\boldsymbol{= \frac{1}{P}\int_0^{[P/T]-1} x\ \ (nT)\exp^{-j\omega_k nT}\ \ T} \tag{6.4}$$

If the sampling interval T is chosen so that it divides the signal period P, then the number of samples under the integral is an integer N = P/T , and we obtain

$$\boldsymbol{X_s(\omega_k) = \frac{T}{P}\sum_{n=0}^{N-1} x\ \ (nT)\exp^{-j\omega_k nT} = \frac{1}{N} DFT_{N,k}(x_p), k=0, \pm 1, \pm 2, \ldots} \tag{6.5}$$

where $x_p = [x(0), x(T), \ldots, x((N - 1)T)]$. Thus, $X_s(\omega_k)$ = $X(\omega_k)$ for all k at which the band limited periodic signal x(t) has a nonzero harmonic. When N is odd, $X(\omega_k)$ can be nonzero for k∈[−(N − 1)/2, (N − 1)/2], while for N even, the maximum nonzero harmonic number range is k∈[−N/2 + 1, N/2 − 1].

Dirac Delta Function and the Fourier Transform

The Dirac delta" function", is defined to serve the purpose which is not a function in the usual sense (mathematicians call it a distribution), through a limiting procedure,

$$\boldsymbol{\delta}(\boldsymbol{x}) = \lim_{\Lambda\to\infty} \boldsymbol{f}_{\Lambda(x)}$$

Where f∧(x) is an ordinary function depending on the parameter ∧.

The requirement on δ(x) is

$$\int \boldsymbol{dxg}(\boldsymbol{x})\boldsymbol{\delta}(\mathbf{x}\text{-}\boldsymbol{a})=\mathbf{g}(\mathbf{a}) \qquad (7.2)$$

Where the range of integration includes the point x=a. The function g is called a "test" function and is supposed to behave "nicely", having no singularities and approaching 0 at infinity. From the basic property (7.2) it follows that

$$\boldsymbol{x\delta}(\boldsymbol{x}) = \mathbf{0}. \qquad (7.3)$$

This is of importance when one divides: From f (x)g(x) = h(x) it does not necessarily follow that f (x) = h(x)/g(x). Rather one has

$$\boldsymbol{f}(\boldsymbol{x}) = \frac{\boldsymbol{h}(\boldsymbol{x})}{\boldsymbol{g}(\boldsymbol{x})} + \boldsymbol{C\delta}(\mathbf{g}(\mathbf{x}))$$

with C an arbitrary constant, as one can see by multiplying both sides by g(x) and using

$$\mathbf{g}(\mathbf{x})\boldsymbol{\delta}(\mathbf{g}(\mathbf{x})) = \mathbf{0}$$

Taking g=1 in (7.2) we get the special result ∫▒〖dxδ(x)〗 = 1. In order to satisfy (7.2) the quantity δ(x) can have support only at x=0, and

to give a finite integral it must be infinite in this point. The ordinary function f_Λ(x) must therefore be more and more peaked around x=0 and become larger and larger for Λ increasing. An important example of a limiting function is

$$\mathbf{f}_{\Lambda}(\mathbf{x}) = \frac{\mathbf{Sin\Lambda x}}{\boldsymbol{\pi}\mathbf{x}} \qquad (7.5)$$

This function has the value Λ/π → ∞ for x=0, and the width of its first peak is π/Λ → 0. For larger values of x it oscillates rapidly. To show that 〖 f〗_Λ approaches a delta function for Λ → ∞, we notice that (take y = Λx

$$\int_{-\infty}^{\infty} \mathbf{dxg}(\mathbf{x}) \frac{\mathbf{Sin\Lambda x}}{\boldsymbol{\pi}\mathbf{x}} = \int_{-\infty}^{\infty} \mathbf{dyg}(\mathbf{x}) \frac{\mathbf{y}}{\Lambda} \frac{\mathbf{Siny}}{\boldsymbol{\pi}\mathbf{y}} \to \frac{\mathbf{g}(\mathbf{0})}{\boldsymbol{\pi}} \int_{-\infty}^{\infty} \frac{\mathbf{dy}}{\mathbf{y}} \mathbf{siny}, \mathbf{for}\ \Lambda \to \infty \qquad (7.6)$$

The property (7.2) is seen to be satisfied (with the arbitrary point a taken to be 0). Therefore

$$\boldsymbol{\delta}(\boldsymbol{x}) = \lim_{\Lambda\to\infty} \frac{\mathbf{Sin\Lambda x}}{\boldsymbol{\pi}\mathbf{x}} = \lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} \frac{\boldsymbol{d\omega}}{\mathbf{2}\boldsymbol{\pi}} \boldsymbol{exp}^{\boldsymbol{\pi\omega x}} \qquad (7.7)$$

There are many other representations of the delta function, but the one given

by(7.7) is especially useful, as seen in the following example. Suppose we have a function f(x) and we then define the new function

$$f(\omega) = \int_{-\infty}^{\infty} dx\, exp^{\pi\omega x} \mathrm{f(x)} \qquad (7.8)$$

This relation can be inverted by the following procedure

$$\boldsymbol{f}(\boldsymbol{x}) = \int_{-\infty}^{\infty} \boldsymbol{dyf}(\boldsymbol{y})\boldsymbol{\delta}(\boldsymbol{x}-\boldsymbol{y}) = \int_{-\infty}^{\infty} \boldsymbol{dyf}(\boldsymbol{y})\boldsymbol{exp}^{\boldsymbol{i\omega(x-y)}} = \int_{-\Lambda}^{\Lambda} \frac{\boldsymbol{d\omega}}{\mathbf{2}\boldsymbol{\pi}} \boldsymbol{exp}^{\boldsymbol{i\omega x}} \boldsymbol{f}(\boldsymbol{\omega}) \qquad (\mathbf{7.9})$$

Where we used (7.2) and (7.7). We then get the pair of equations in the limit ∧→∞

$$\boldsymbol{f}(\boldsymbol{x}) = \int_{-\infty}^{\infty} \frac{\boldsymbol{d\omega}}{\mathbf{2}\boldsymbol{\pi}} \boldsymbol{exp}^{\boldsymbol{i\omega x}} \boldsymbol{f}(\boldsymbol{\omega}), \boldsymbol{f}(\boldsymbol{\omega}) = \int_{-\infty}^{\infty} \frac{\boldsymbol{d\omega}}{\mathbf{2}\boldsymbol{\pi}} \boldsymbol{exp}^{\boldsymbol{i\omega x}} \boldsymbol{f}(\boldsymbol{x}) \qquad (\mathbf{7.10})$$

Thus,(7.9) constitutes a one-line derivation. This shows the enormous power in using the delta function.

It should be noticed that because of (7.7) one has the following formal integral representation of the delta function,

$$\boldsymbol{\delta(x)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}exp^{i\omega x}=\int_{0}^{\infty}\frac{d\omega}{2\pi}cos(\omega x)}. \qquad (7.11)$$

This could also be obtained by taking f (x) = δ(x) in (7.10). The limits of the integral should really be understood as being between −Λ and Λ, but in calculations one can often use this integral with Λ already taken to infinity. As a further illustration of the delta function, let us return to the Fourier series discussed in the former section. Taking into account that S_∞(θ) = f (θ) (assuming f to be continuous),

$$\boldsymbol{\delta(x)=\frac{1}{\pi}\lim_{N\to\infty}[\frac{1}{2}+\sum_{n=1}^{N}cosnx]=\frac{1}{2\pi}\lim_{N\to\infty}\sum_{n=-N}^{N}exp^{inx}=\lim_{N\to\infty}\frac{\sin(N+\frac{1}{2})x}{2\pi\sin(\frac{1}{2})x}} \qquad (7.12)$$

This representation of the delta function is similar to the one exhibition in (7.7) and in (7.11. In particular we can write (7.12) formally as

$$\boldsymbol{\delta(x)=\frac{1}{\pi}\lim_{N\to\infty}[\frac{1}{2}+\sum_{n=1}^{N}cosnx]=\frac{1}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}cos(nx)} \qquad \textbf{(7.13)}$$

which is a discrete version of (7.11). To show (7.12) directly we study how it acts in an integral,

$$\boldsymbol{\int_{-\infty}^{\infty}f(x)\frac{\sin(N+\frac{1}{2})x}{2\pi\sin(\frac{1}{2})x}dx=\int_{-\infty}^{\infty}f(\frac{u}{N+\frac{1}{2}})\frac{sinu}{2\pi(N+\frac{1}{2})}sin(\frac{u}{2}(N+\frac{1}{2}))du\to f(0)} \qquad \textbf{(7.14)}$$

$$\boldsymbol{\int_{-\infty}^{\infty}\frac{sinu}{\pi u}du=f(0)} \qquad \textbf{(7.15)}$$

For N → ∞. The last expression is the same as we would have obtained by use of (7.5). So under the integral sign the two representations (7.5) and (7.9) in the end

become the same. The sine function serves as a demonstration of the necessity of complex numbers in the Fourier transform; a real-valued sine wave is described by a completely imaginary frequency representation.

SPECTROGRAMS AND CHROMAGRAMS

The Short-Time Fourier Transform (STFT) of a small-time interval of the input signal

of frequencies as time progresses (Smith,2007) can be constructed. The STFT allows us to add a time change over time. Since frequencies are representative of pitches, we can use the signal. The creation of these spectrographs is crucial in determining how chords are changing in music. As stated previously, the octave information of. the sound is irrelevant. We determine the chroma intensity by collecting all intensities of a note regardless of its octave. Let us look at an example. We will use the MATLAB code by Ellis. A piano playing an ascending chromatic scale.

CONCLUSION

Analog signal is discretized to enable computer analysis of digital signals. In this paper we develop the discrete Fourier transform from can be used to analyze a musical signal for chord structure the mathematics utilized in digital signal processing. In this paper we have laid a foundation for understanding the mathematics behind a chord recognition model. We have provided the reader with. chord recognition. In addition, we derived the DFT from the Fourier transform and gained an appreciation for the applications of Fourier analysis in signal processing. Finally, we have shown through examples how the FFT of rhythm, pitch, and harmony. These rules and the patterns they create entice to identify played chords within a music file.

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